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ON A CLASS OF SINGULAR PSEUDO-
DIFFERENTIAL OPERATORS

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In the past one or two decades there appeared extensive works on singular partial differential operators, see e.g. S. Alinhac [1],[2] and H. Tahara [13], [14], [15]. Among them the so-called Fuchsian type equations are most remarkable, see Baouendi and Goulaouic [4] and the author's paper [5]. But in the classical work of J. Hadamard on the construction of the fundamental solution, there appears non-Fuchsian partial differential equation whose properties are quite different from that of the Fuchsian type equations, see [8], [5], [6]. Hence it is desirable to study in more detail a class of singular pseudo-differential operators of the form

$$D_t u + A(x, t, D_x) u + t^{-m} B(x, D_x) u = f, \quad m \in \mathbb{N} \quad (1)$$

where A, B are proper pseudo-differential operators of first order on a C^∞ manifold M (countable at infinity, t a parameter) with complete symbols $a(x, t, \xi)$ and $b(x, \xi)$. $b(x, \xi)$ is positively homogeneous in ξ of degree 1.

Our main result is, under certain conditions, equation (1) may be reduced to

$$D_t V + A(x, t, D_x) V = F. \quad (2)$$

Now, let's explain our chief idea, but for the moment only formally.

Consider the equation

$$t^m D_t u + B(x, D_x)u = v \quad m \text{ even} \quad (3)$$

and the diffeomorphism $R_t \setminus 0 \rightarrow R_\tau \setminus 0$

$$\tau = t^{1-m} / (1-m), \quad (4)$$

or the equation pair

$$\left\{ \begin{array}{ll} t^m D_t u + B(x, D_x)u = v & t > 0 \\ -t^m D_t u + B(x, D_x)u = v & t < 0 \end{array} \right. \quad m \text{ odd}, m > 1 \quad (5)$$

and the diffeomorphism $R_t \setminus 0 \rightarrow R_\tau \setminus 0$

$$\left\{ \begin{array}{ll} \tau = t^{1-m} / (1-m) & t > 0 \\ \tau = -(-t)^{1-m} / (1-m) & t < 0, \end{array} \right. \quad (6)$$

or when $m=1$ consider separately in $R_t^+ : t > 0$ and $R_t^- : t < 0$ the equation

$$t D_t u + B(x, D_x)u = v \quad (7)$$

and the diffeomorphism $R_t^+ \setminus 0 \rightarrow R_\tau \setminus 0$

$$\left\{ \begin{array}{ll} \tau = \ln t & t > 0 \\ \tau = \ln(-t) & t < 0, \end{array} \right. \quad (8)$$

we shall have

$$D_\tau u + B(x, D_x)u = v, \quad (9)$$

hence formally

$$u = \exp(-i\tau B)v. \quad (10)$$

Substituting it into (1) gives

$$\frac{\partial v}{\partial \tau} + i \exp(i\tau B) \cdot A \cdot \exp(-i\tau B)v = \exp(i\tau B)f.$$

when A and B commute, hence A and $\exp(i\tau B)$ also commute, we have

$$\frac{\partial v}{\partial \tau} + iAv = F. \quad (11)$$

(11) is an equation without singularity in the left hand side.

It is readily seen that the arguments above may be made rigorous once $\{\exp(-i\tau B)\}$ ($\forall \tau \in \mathbb{R}$) is defined rigorously and proved to be a group. This can be done when the manifold M is compact without boundary and $B(x, D_x)$ formally self-adjoint and elliptic as is well known in the spectral theory of self-adjoint operators. But there is another approach to this aim, i.e. to solve the Cauchy problem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial \tau} + iBu = 0 \\ u(0, x) = u_0(x). \end{array} \right. \quad (12)$$

In fact, by "Fourier method", it is easy to see, the solution of Cauchy problem (12) is just $u = \exp(-i\tau B)u_0$.

But, the Cauchy problem is solvable not only when B is formally self-adjoint and elliptic. Hence, if we can find the fundamental solution or at least the parametrix $U(\tau)$ of the Cauchy problem (12), its solution would just be

$$u(\tau, x) = U(\tau)u_0(x)$$

and we may use $U(\tau)$ as the exponential $\exp(-i\tau B)$.

In defining the parametrix of the Cauchy problem (12), behavior of the orbits of the Hamiltonian field

$$H_B = (b\xi(x, \zeta)\partial_x, -b_x(x, \zeta)\partial_\xi)$$

is very important. In our case, we should at least assume its existence for all $\tau \in \mathbb{R}_\tau$ since τ near 0 corresponds to τ near infinity.

Now, we proceed to construct the parametrix for the Cauchy problem (12) first for small τ . Such construction, as usual, amounts to seek an operator $U(\tau): C_0^\infty(M) \rightarrow C^\infty(M)$ such that

$$\begin{cases} (D_\tau + B) U(\tau) \in S^{-\infty} \\ U(0) - 1 \in S^{-\infty} \end{cases} \quad (13)$$

Linearity of the problem allows us to use a partition of unity and reduce our problem to the case $M = \mathbb{R}^n$, i.e., to consider (13) in $\mathbb{R}_\tau \times \mathbb{R}_x^n$.

Now assume $U(\tau)$ to be a Fourier integral operator (F.I.O.)

$$U(\tau)f(x) = (2\pi)^{-n} \int \exp[iS(\tau, x, y, n)] q(\tau, x, n) f(y) dy dn \quad (14)$$

$$f(x) \in C_0^\infty(\mathbb{R}^n)$$

with a distribution kernel

$$U(\tau, x, y) = (2\pi)^{-n} \int \exp [iS(\tau, x, y, \eta)] q(\tau, x, \eta) d\eta.$$

Let $S(\tau, x, y, \eta) = \Phi(\tau, x, \eta) - y \cdot \eta$, we have the following Cauchy problem for $\Phi(\tau, x, \eta)$

$$\left\{ \begin{array}{l} \frac{\partial \Phi}{\partial \tau} + B(x, \Phi_x) = 0 \\ \Phi(0, x, \eta) = x \cdot \eta. \end{array} \right. \quad (15)$$

In order to solve it, we turn to the Hamiltonian system

$$\frac{dx}{ds} = b_\eta(x, \eta), \quad \frac{d\eta}{ds} = -b_x(x, \eta). \quad (16)$$

Assuming the initial conditions

$$x|_{\tau=0} = x_0(z_1, \dots, z_n), \quad \eta|_{\tau=0} = \eta_0(z_1, \dots, z_n)$$

where η_1, \dots, η_n are n parameters such that

$$\det \left(\frac{\partial x}{\partial z} \right)_{\tau=0} \neq 0,$$

we have solution for (16) as

$$x = x(\tau, z), \quad \eta = \eta(\tau, z) \quad z = (z_1, \dots, z_n) \quad (17)$$

From classical theory of partial differential equations of first order, we have

$$\begin{aligned} \Phi(\tau, x) = (x_0 \cdot \eta_0)(z) + \int_0^\tau [\langle \eta(\rho, z), \frac{dx(\rho, z)}{d\rho} \rangle - \\ - B(x(\rho, z), \eta(\rho, z))] d\rho, \end{aligned} \quad (18)$$

which is valid as far as $\det \frac{\partial(\tau, x)}{\partial(\tau, z)} = \det \left(\frac{\partial x}{\partial z} \right) \neq 0$, i.e., for $|\tau|$ sufficiently small.

The amplitude $q(\tau, x, \eta)$ is to be found in the class

$$q(\tau, x, \eta) \sim \sum_{k=0}^{\infty} q_k(\tau, x, \eta), \quad (19)$$

where $q_k(\tau, x, \eta)$ are positively homogeneous in η of degree $-k$.

For q_k we have the so-called transport equations

$$\partial_{\tau} q_0 + \sum_{|\alpha|=1} b^{(\alpha)}(x, \phi_x) \partial_x^{\alpha} q_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!} b^{(\alpha)}(x, \phi_x) \partial_x^{\alpha} \phi \cdot q_0 = 0,$$

$$\partial_{\tau} q_k + \sum_{|\alpha|=1} b^{(\alpha)}(x, \phi_x) \partial_x^{\alpha} q_k + \sum_{|\alpha|=2} \frac{1}{\alpha!} b^{(\alpha)}(x, \phi_x) \partial_x^{\alpha} \phi \cdot q_k \quad (20)$$

$$+ R_k(q_0, \dots, q_{k-1}) = 0, \quad k \geq 1,$$

$$q_k(0, x, \eta) = \delta_{0k}. \quad (21)$$

Hence

$$q_0(\tau, x, \eta) = \sqrt{\left| \frac{J(0, z)}{J(\tau, z)} \right|} \exp \left[\int_0^{\tau} \frac{1}{2} \text{tr} \left(\frac{\partial^2 b(x, \phi_x)}{\partial x \partial \eta} \right) d\tau \right] \quad (22)$$

Where $J(\tau, z) = \det \frac{\partial(\tau, x)}{\partial(\tau, z)}$. Similar results hold for q_k when $k \geq 1$.

Thus (22) holds only when $\det \frac{\partial(\tau, x)}{\partial(\tau, z)} = \det \left(\frac{\partial x}{\partial z} \right) \neq 0$,

i.e., when $|\tau|$ sufficiently small.

We shall now follow Maslov's line to construct a parametrix valid for all τ . The intrinsic object connected with the Cauchy problem (15) is a Lagrangean manifold Λ^{n+1} of dimension $n+1$ constructed in the following way. Through every point of an n -dimensional isotropic manifold $\Lambda^n \subset T^k(\mathbb{R}_{\tau} \times M) \setminus 0 = (\mathbb{R}_{\tau} \times \mathbb{R}_x^n) \times [(\mathbb{R}_E \times \mathbb{R}_p^n) \setminus 0]$ passes an orbit of the Hamiltonian vector field $H_B(x, p)$

$$\frac{d\tau}{ds} = 1, \quad \frac{dE}{ds} = 0$$

$$\frac{dx}{ds} = b_p(x,p), \quad \frac{dp}{ds} = -b_x(x,p)$$

these orbits then form Λ^{n+1} . Here $\Lambda^n \subset \{(\tau, x, E, p) : E + b(x, p) = 0\}$, E is the dual variable of τ . Later τ and E will be denoted by x_0 and p_0 respectively.

Maslov and Arnold [11], [13] proved that there exists a canonical atlas for Λ^{n+1} , such that the local coordinate in any chart (simply connected) should be of the form (x_I, p_J) , where I and J are subsets of $\{0, 1, \dots, n\}$. $I \cup J = \{0, 1, \dots, n\}$, $I \cap J = \emptyset$. Those charts which are diffeomorphic to \mathbb{R}_x are called regular. Those wherein is a point with no neighborhood diffeomorphic to \mathbb{R}_x are called singular charts and such points singular points. The set of singular points $\Sigma(\Lambda^{n+1})$ is a cycle — singular cycle, which can always be assumed to be an n -dimensional submanifold of Λ^{n+1} .

In Maslov's work, x and p are put on a completely equal footing, hence we should modify the class of symbols of pseudo-differential operators in accordance.

Definition Let $a(x, p) \in C^\infty(\mathbb{R}_x \times \mathbb{R}_p)$ satisfy the following condition :

There exists constant $C_{\alpha\beta} > 0$ such that for all $(x, p) \in \mathbb{R}_x \times \mathbb{R}_p$

$$|\partial_x^\alpha \partial_p^\beta a(x, p)| \leq C_{\alpha\beta} (1 + |x|)^{m-|\alpha|} (1 + |p|)^{m-|\beta|} \quad (23)$$

we say the symbol $a(x, p) \in M^m$.

Now let there exists a C^∞ positive 1-density ds on M , which is invariant under the action of H_B . Our main result is

Theorem 1 If M is a C^∞ (countable at infinity) manifold of dimension n with a C^∞ positive 1-density ds defined on it. Let $B(x, D_x)$ be a proper pseudo-differential operator of order 1 with a real symbol $b(x, p) \in M^1$ which is positively homogeneous in (x_I, p_J) of degree 1 for arbitrary complementary subsets I and J of $\{1, 2, \dots, n\}$ with $I \cap J = \emptyset$. Assume $H_B = (b_p(x, p) \partial_x, -b_x(x, p) \partial_p)$ defines a 1-parameter group of diffeomorphism and ds is invariant under it. Denote by Λ^{n+1} the Lagrangean manifold defined by the equation (15) with canonical coordinates (x, p) . Suppose the following conditions of quantization hold

$$(1) \quad \int_\gamma p dx = 0 \quad \text{for every closed path } \gamma \text{ on } \Lambda^{n+1}; \quad (24)$$

$$(2) \quad \text{The Maslov index on every cycle is 0.} \quad (25)$$

Under these conditions, a parametrix for arbitrary τ for the Cauchy problem (12) exists.

Outline of Proof. Let Ω_0 be a regular chart where

$$ds = |dx|.$$

The phase function $\Phi(\tau, x, \eta)$ in (14) where $S_{(\tau, x, y, \eta)} = \Phi(\tau, x, \eta) - y \cdot \eta$ is just the generating function of the Lagrangean Manifold Λ^{n+1} in this coordinate patch, which may be expressed as

$$\Phi_{\Omega_0}(\tau, x, \eta) = \int_{P_0}^P p dx \quad (26)$$

where P_0 is a fixed point in Ω_0 and $P = (\tau, x, \eta)$. In Maslov's work [11], a pre-canonical operator on Ω_0

$$K_{\Omega_0}[q_{\Omega_0}(\tau, x, \eta)] = q_{\Omega_0}(\tau, x, \eta) \exp \left[i \int_{P_0}^P p dx \right] \sqrt{\frac{d\sigma}{|dx|}} \quad (27)$$

is defined. Here $q_{\Omega_0}(\tau, x, \eta)$ is just $q(\tau, x, \eta)$ constructed above. Thus the local parametrix (14) may be written as

$$[U_{\Omega_0}(\tau)f](Q) = (2\pi)^{-n} \int K_{\Omega_0}[q_{\Omega_0}(\tau, Q, \eta)] \hat{f}(\eta) d\eta, \quad (28)$$

$$Q = Q(x).$$

If we follow the orbit of H_B and enter another coordinate patch Ω_1 of Λ^{n+1} with focal coordinate (x_I, p_J) , denote the generating function of Λ^{n+1} on Ω_1 by Φ_{Ω_1}

$$\Phi_{\Omega_1}(\tau, Q, \eta) = \int_{Q_0}^Q p dx - \langle x_J(x_I, p_J), p_J \rangle,$$

$$Q_0 \in \Omega_0 \cap \Omega_1$$

The pre-canonical operator K_{Ω_1} is

$$K_{\Omega_1}[q(\tau, Q, \eta)] = \exp \left(\frac{\pi i}{2} \alpha_1 \right) q_{\Omega_1}(\tau, Q, \eta) \cdot$$

$$\exp [i \Phi_{\Omega_1}(\tau, Q, \eta)] \sqrt{\frac{d\sigma}{|dx_I dp_J|}}$$

$$Q = Q(x_I, p_J)$$

$$\alpha_1 = \text{Maslov's index of } \Omega_1$$

and we try to find a local parametrix $U_{\Omega_1}(\tau)$ of the Cauchy problem (13) as

$$[U_{\Omega_1}(\tau)f](Q) = (2\pi)^{-n} F_{p_J \rightarrow x_J}^{-1} \int_{K_{\Omega_1}} [q_{\Omega_1}(\tau, Q, \eta)] \hat{f}(\eta) d\eta, \quad (29)$$

$$Q = Q(x_I, p_J),$$

where $F_{p_J \rightarrow x_J}^{-1}$ is the inverse partial Fourier transform.

When (29) is substituted into equation (13), we shall have corresponding transport equation for $q_{\Omega_1}(\tau, Q, \eta)$, but the

initial condition will no longer be (21), but with the values at $Q_0 \in \Omega_0 \cap \Omega_1$ of $q_{\Omega_0 k}(\tau, Q, \eta)$ as the initial values of $q_{\Omega_1 k}(\tau, Q, \eta)$.

It is known [11], at $Q = Q(x) = Q(x_I, p_J) \in \Omega_0 \cap \Omega_1$,

$$[U_{\Omega_0}(\tau)f](Q) = [U_{\Omega_1}(\tau)f](Q). \quad (30)$$

Thus, we may piece together the local parametrices obtained on each coordinate patch as follows. Let $\{e_i\}$ be a partition of unity subordinated to the canonical atlas $\{\Omega_j\}$, define

$$q(\tau, Q, \eta) = \sum_j q_j(\tau, Q, \eta) = \sum_j e_j q_{\Omega_j}(\tau, Q, \eta),$$

$$K_\Lambda[q(\tau, Q, \eta)] = \sum_j F_{p_{J_j} \rightarrow x_{J_j}}^{-1} K_{\Omega_j}[e_j q_{\Omega_j}(\tau, Q, \eta)],$$

(Maslov's global cononical operator).

The parametrix for arbitrary τ will be

$$[U(\tau)f]Q = (2\pi)^{-n} \int K_{\Lambda}[q(\tau, Q, \eta)] \hat{f}(\eta) d\eta. \quad (31)$$

In order (31) be well-defined, it is sufficient that

$$(i) \quad \int_{\gamma} p dx = 0 \quad \text{on every cycle } \gamma;$$

(ii) The Maslov's index for every cycle is 0.

Q. E. D.

Now, it is easy to prove the following properties of $U(\tau)$.

Theorem 2 If the Cauchy problem (13) has a unique (up to mod $S^{-\infty}$) solution $U(\tau)$, then

$$U(\tau_1 + \tau_2) = U(\tau_1) \circ U(\tau_2), \quad (\text{mod } S^{-\infty}). \quad (32)$$

Theorem 3 If $A(x, D_x)$ is independent of τ and $[A, B] = 0$, then $[A, U] = 0$. Equalities are in a mod $S^{-\infty}$ sense.

The proofs are omitted.

Now we turn to some concrete cases.

First let M be a compact C^∞ manifold (countable at infinity) without boundary, $B(x, D_x)$ formally self-adjoint and elliptic. In this case $\{\exp(i\tau B)\}$ may be defined by spectral-theoretic method and is a group of unitary operators. It is easy to prove

Theorem 4 If A and B commute, A and $U(\tau)$ also commute (both in a mod $S^{-\infty}$ sense)

Proof is omitted.

Turn to the equation (1). Introduce the new unknown v by

$$u = \exp(-i\tau B)v(\tau, x) = U(\tau)v(\tau, x) \pmod{S^{-\infty}}, \quad (33)$$

for $v(\tau, x)$ we have

$$D_t v + U^{-1}A(t, x, D_x)Uv = U^{-1}f \pmod{S^{-\infty}}. \quad (34)$$

Theorem 5 If A and B commute, equation (1) may be reduced to

$$D_t v + Av = U^{-1}f \quad (35)$$

If $[A, B] \neq 0$, but $[[A, B], B] = 0$, equation (1) may be reduced to

$$D_t v + Av + \frac{i[A, B]}{(1-m)t^{m-1}} v = U^{-1}f \quad m > 1, \quad (36)$$

$$D_t v + Av + i\ln t \cdot [A, B]v = U^{-1}f \quad m=1,$$

With lower order singularity. All the equalities are taken in $\text{mod } S^{-\infty}$ sense.

The proof is also straightforward.

In the second case, M is no longer assumed to be compact, but the symbol $b(x, \xi)$ of $B(x, D_x)$ is assumed to be real and the Hamiltonian of $b(x, \xi)$ is not the radial direction, i.e., H_B and $\xi \partial_\xi$ are not parallel. A typical example of this case is

$$t^m (D_t + A(t, x, D_x))u + D_{x_1} u = t^m f. \quad (37)$$

We may use the method above or the partial Fourier transformation to construct its parametrix, but the most straightforward way is to introduce new variables, e.g., when $m > 1$,

$$\tilde{x}_1 = x_1 - t^{1-m} / (1-m), \quad \tilde{t} = t, \quad \tilde{x}_j = x_j \quad j > 1 \quad (38)$$

Then we have (still use the notations t and x)

$$\begin{aligned} D_t u + A(t, x_1 + t^{1-m}/(1-m), x_2, \dots, x_n, D_x) u \\ = F(t, x_1 + t^{1-m}/(1-m), x_2, \dots, x_n) \end{aligned} \quad (39)$$

In fact, in this case $B = D_{x_1} = \frac{1}{i} \partial_{x_1}$ is still a self-adjoint operator and its exponential is just the translation operator. What is more important is that we may use a canonical transformation such that microlocally equation (1) is equivalent to (37).

Theorem 6 Under the conditions above, equation (1) may be reduced to an equation without singularity when $[A, B] = 0 \pmod{S^{-\infty}}$, or reduced to an equation with lower singularity when $[[A, B], B] = 0 \pmod{S^{-\infty}}$.

Proof It is well known [7], [9] that there exists a unitary Fourier integral operator U such that microlocally

$$U^{-1} B U = D_{x_1} \pmod{S^{-\infty}}$$

Denote by $A_1(t, x, D_x) = U^{-1} A U$, we see

$$0 = U^{-1} [A, B] U = [A_1, D_{x_1}]$$

when $[A, B] = 0$. By computing the symbols, we have

$$D_{x_1} a_1(t, x, \xi) = 0, \quad (\text{mod } S^{-\infty}),$$

where $a_1(t, x, \xi)$ is the symbol of A_1 , hence a_1 is independent of x_1 . By introducing the new variables (38), we see equation (39) becomes

$$D_t u + A_1(t, x, D_x) u = f(t, x_1 + t^{1-m}/(1-m), x_2, \dots, x_n). \quad (40)$$

When $m=1$, corresponding result will be obtained.

When $[[A, B], B] = 0$, it is easy to see

$$A(t, x, D_x) = A_1(t, x, D_x) + x_1 A_2(t, x, D_x) \quad (\text{mod } S^{-\infty}),$$

where A_1 and A_2 have symbols independent of x_1 . Use the method above, we have

$$\begin{aligned} D_t u + A_1(t, x, D_x) u + [x_1 + t^{1-m}/(1-m)] A_2(x, t, D_x) u \\ = f(t, x_1 + t^{1-m}/(1-m), x_2, \dots, x_n) \end{aligned} \quad (41)$$

The reduction above holds microlocally, but since it holds in every conic neighborhood of any point in T^*M , we can use it at every point of M .

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